

# UNIQUENESS IN AN INVERSE BOUNDARY PROBLEM FOR A MAGNETIC SCHRÖDINGER OPERATOR WITH A BOUNDED MAGNETIC POTENTIAL

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**ABSTRACT.** We show that the knowledge of the set of the Cauchy data on the boundary of a bounded open set in  $\mathbb{R}^n$ ,  $n \geq 3$ , for the magnetic Schrödinger operator with  $L^\infty$  magnetic and electric potentials determines the magnetic field and electric potential inside the set uniquely. The proof is based on a Carleman estimate for the magnetic Schrödinger operator with a gain of two derivatives.

## 1. INTRODUCTION AND STATEMENT OF RESULT

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open set, and let  $u \in C_0^\infty(\Omega)$ . We consider the magnetic Schrödinger operator,

$$\begin{aligned} L_{A,q}(x, D)u(x) &:= \sum_{j=1}^n (D_j + A_j(x))^2 u(x) + q(x)u(x) \\ &= -\Delta u(x) + A(x) \cdot Du(x) + D \cdot (A(x)u(x)) + ((A(x))^2 + q(x))u(x), \end{aligned}$$

where  $D = i^{-1}\nabla$ ,  $A \in L^\infty(\Omega, \mathbb{C}^n)$  is the magnetic potential, and  $q \in L^\infty(\Omega, \mathbb{C})$  is the electric potential. We have  $Au \in L^\infty(\Omega, \mathbb{C}^n) \cap \mathcal{E}'(\Omega, \mathbb{C}^n)$ , and therefore,

$$L_{A,q} : C_0^\infty(\Omega) \rightarrow H^{-1}(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$$

is a bounded operator. Here  $\mathcal{E}'(\Omega) = \{v \in \mathcal{D}'(\Omega) : \text{supp } (v) \text{ is compact}\}$ .

Let us now introduce the Cauchy data for an  $H^1(\Omega)$  solution  $u$  to the equation

$$L_{A,q}u = 0 \quad \text{in } \Omega, \tag{1.1}$$

in the sense of distributions. First, following [1, 17], we define the trace space of the space  $H^1(\Omega)$  as the quotient space  $H^1(\Omega)/H_0^1(\Omega)$ . The associated trace map  $T : H^1(\Omega) \rightarrow H^1(\Omega)/H_0^1(\Omega)$ ,  $Tu = [u]$ , is the quotient map. Here  $H_0^1(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  with respect to the  $H^1(\Omega)$ -topology.

Notice that if  $\Omega$  has a Lipschitz boundary, then the space  $H^1(\Omega)/H_0^1(\Omega)$  can be naturally identified with the Sobolev space  $H^{1/2}(\partial\Omega)$ . Indeed, in this case the kernel of the continuous surjective map  $H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ ,  $u \mapsto u|_{\partial\Omega}$  is precisely  $H_0^1(\Omega)$ , see [12, Theorems 3.37 and 3.40].

For  $u \in H^1(\Omega)$  satisfying (1.1), we can define  $N_{A,q}u$ , formally given by  $N_{A,q}u = (\partial_\nu u + i(A \cdot \nu)u)|_{\partial\Omega}$ , as an element of the dual space  $(H^1(\Omega)/H_0^1(\Omega))'$  as follows. For  $[g] \in H^1(\Omega)/H_0^1(\Omega)$ , we set

$$(N_{A,q}u, [g])_\Omega := \int_\Omega (\nabla u \cdot \nabla g + iA \cdot (u\nabla g - g\nabla u) + (A^2 + q)ug) dx. \quad (1.2)$$

As  $u$  is a solution to (1.1),  $N_{A,q}u$  is a well-defined element of  $(H^1(\Omega)/H_0^1(\Omega))'$ .

We define the set of the Cauchy data for solutions of the magnetic Schrödinger equation as follows,

$$C_{A,q} := \{(Tu, N_{A,q}u) : u \in H^1(\Omega) \text{ and } L_{A,q}u = 0 \text{ in } \Omega\}.$$

The inverse boundary value problem for the magnetic Schrödinger operator  $L_{A,q}$  is to determine  $A$  and  $q$  in  $\Omega$  from the set of the Cauchy data  $C_{A,q}$ .

Similarly to [20], there is an obstruction to uniqueness in this problem given by the following gauge equivalence of the set of the Cauchy data: if  $\psi \in W^{1,\infty}$  in a neighborhood of  $\bar{\Omega}$  and  $\psi|_{\partial\Omega} = 0$ , then  $C_{A,q} = C_{A+\nabla\psi,q}$ , see Lemma 3.1 below. Hence, the map  $A \mapsto A + \nabla\psi$  transforms the magnetic potential into a gauge equivalent one but preserves the induced magnetic field  $dA$ , which is defined by

$$dA = \sum_{1 \leq j < k \leq n} (\partial_{x_j} A_k - \partial_{x_k} A_j) dx_j \wedge dx_k,$$

in the sense of distributions. Here  $A = (A_1, \dots, A_n)$ . In view of this and of the fact that the magnetic field is a physically observable quantity, one may hope to recover the magnetic field  $dA$  and the electric potential  $q$  in  $\Omega$  from the set of the Cauchy data  $C_{A,q}$ .

As it has been shown by several authors, the knowledge of the set of the Cauchy data  $C_{A,q}$  for the magnetic Schrödinger operator  $L_{A,q}$  does determine the magnetic field  $dA$  and the electric potential  $q$  in  $\Omega$  uniquely, under certain regularity assumptions on  $A$  and  $q$ . In [20], this result was established for magnetic potentials in  $W^{2,\infty}$ , satisfying a smallness condition, and  $L^\infty$  electric potentials. In [13], the smallness condition was eliminated for smooth magnetic and electric potentials, and for compactly supported  $C^2$  magnetic potentials and  $L^\infty$  electric potentials. The uniqueness results were subsequently extended to  $C^1$  magnetic potentials in [22], to some less regular but small potentials in [14], and to Dini continuous magnetic potentials in [17].

The purpose of this paper is to extend the uniqueness result to the case of magnetic Schrödinger operators with magnetic potentials that are of class  $L^\infty$ . Our main result is as follows.

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open set, and let  $A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n)$  and  $q_1, q_2 \in L^\infty(\Omega, \mathbb{C})$ . If  $C_{A_1, q_1} = C_{A_2, q_2}$ , then  $dA_1 = dA_2$  and  $q_1 = q_2$  in  $\Omega$ .*

Notice in particular that in Theorem 1.1 no regularity assumptions on the boundary of  $\Omega$  are required.

The key ingredient in the proof of Theorem 1.1 is a construction of complex geometric optics solutions for the magnetic Schrödinger operator  $L_{A,q}$  with  $A \in L^\infty(\Omega, \mathbb{C}^n)$  and  $q \in L^\infty(\Omega, \mathbb{C})$ . When constructing such solutions, we shall first derive a Carleman estimate for the magnetic Schrödinger operator  $L_{A,q}$ , with a gain of two derivatives, which is based on the corresponding Carleman estimate for the Laplacian, obtained in [19]. Another crucial observation, which allows us to handle the case of  $L^\infty$  magnetic potentials is that it is in fact sufficient to approximate the magnetic potential by a sequence of smooth vector fields, in the  $L^2$  sense.

We would also like to mention that another important inverse boundary value problem, for which the issues of regularity have been studied extensively, is Calderón's problem for the conductivity equation, see [4]. The unique identifiability of  $C^2$  conductivities from boundary measurements was established in [21]. The regularity assumptions were relaxed to conductivities having  $3/2 + \varepsilon$  derivatives in [2], and the uniqueness for conductivities having exactly  $3/2$  derivatives was obtained in [15], see also [3]. In [8], uniqueness for conormal conductivities in  $C^{1+\varepsilon}$  was shown. The recent work [9] proves a uniqueness result for Calderón's problem with conductivities of class  $C^1$  and with Lipschitz continuous conductivities, which are close to the identity in a suitable sense.

The paper is organized as follows. Section 2 contains the construction of complex geometric optics solutions for the magnetic Schrödinger operator with  $L^\infty$  magnetic and electric potentials. The proof of Theorem 1.1 is then completed in Section 3.

## 2. CONSTRUCTION OF COMPLEX GEOMETRIC OPTICS SOLUTIONS

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open set. Following [5, 11], we shall use the method of Carleman estimates to construct complex geometric optics solutions for the magnetic Schrödinger equation  $L_{A,q}u = 0$  in  $\Omega$ , with  $A \in L^\infty(\Omega, \mathbb{C}^n)$  and  $q \in L^\infty(\Omega, \mathbb{C})$ .

Let us start by recalling the Carleman estimate for the semiclassical Laplace operator  $-h^2\Delta$  with a gain of two derivatives, established in [19], see also [11]. Here  $h > 0$  is a small semiclassical parameter. Let  $\tilde{\Omega}$  be an open set in  $\mathbb{R}^n$  such that  $\Omega \subset\subset \tilde{\Omega}$  and let  $\varphi \in C^\infty(\tilde{\Omega}, \mathbb{R})$ . Consider the conjugated operator

$$P_\varphi = e^{\frac{\varphi}{h}}(-h^2\Delta)e^{-\frac{\varphi}{h}},$$

with the semiclassical principal symbol

$$p_\varphi(x, \xi) = \xi^2 + 2i\nabla\varphi \cdot \xi - |\nabla\varphi|^2, \quad x \in \tilde{\Omega}, \quad \xi \in \mathbb{R}^n.$$

We have for  $(x, \xi) \in \bar{\Omega} \times \mathbb{R}^n$ ,  $|\xi| \geq C \gg 1$ , that  $|p_\varphi(x, \xi)| \sim |\xi|^2$  so that  $P_\varphi$  is elliptic at infinity, in the semiclassical sense. Following [11], we say that  $\varphi$  is a limiting Carleman weight for  $-h^2\Delta$  in  $\tilde{\Omega}$ , if  $\nabla\varphi \neq 0$  in  $\tilde{\Omega}$  and the Poisson bracket of  $\text{Re } p_\varphi$  and  $\text{Im } p_\varphi$  satisfies,

$$\{\text{Re } p_\varphi, \text{Im } p_\varphi\}(x, \xi) = 0 \quad \text{when} \quad p_\varphi(x, \xi) = 0, \quad (x, \xi) \in \tilde{\Omega} \times \mathbb{R}^n.$$

Examples of limiting Carleman weights are linear weights  $\varphi(x) = \alpha \cdot x$ ,  $\alpha \in \mathbb{R}^n$ ,  $|\alpha| = 1$ , and logarithmic weights  $\varphi(x) = \log|x - x_0|$ , with  $x_0 \notin \tilde{\Omega}$ . In this paper we shall only use the linear weights.

Our starting point is the following result due to [19].

**Proposition 2.1.** *Let  $\varphi$  be a limiting Carleman weight for the semiclassical Laplacian on  $\tilde{\Omega}$ , and let  $\varphi_\varepsilon = \varphi + \frac{h}{2\varepsilon}\varphi^2$ . Then for  $0 < h \ll \varepsilon \ll 1$  and  $s \in \mathbb{R}$ , we have*

$$\frac{h}{\sqrt{\varepsilon}} \|u\|_{H_{\text{scl}}^{s+2}(\mathbb{R}^n)} \leq C \|e^{\varphi_\varepsilon/h}(-h^2\Delta)e^{-\varphi_\varepsilon/h}u\|_{H_{\text{scl}}^s(\mathbb{R}^n)}, \quad C > 0, \quad (2.1)$$

for all  $u \in C_0^\infty(\Omega)$ .

Here

$$\|u\|_{H_{\text{scl}}^s(\mathbb{R}^n)} = \|\langle hD \rangle^s u\|_{L^2(\mathbb{R}^n)}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2},$$

is the natural semiclassical norm in the Sobolev space  $H^s(\mathbb{R}^n)$ ,  $s \in \mathbb{R}$ .

Next we shall derive a Carleman estimate for the magnetic Schrödinger operator  $L_{A,q}$  with  $A \in L^\infty(\Omega, \mathbb{C}^n)$  and  $q \in L^\infty(\Omega, \mathbb{C})$ . To that end we shall use the estimate (2.1) with  $s = -1$ , and with  $\varepsilon > 0$  being sufficiently small but fixed, i.e. independent of  $h$ . We have the following result.

**Proposition 2.2.** *Let  $\varphi \in C^\infty(\tilde{\Omega}, \mathbb{R})$  be a limiting Carleman weight for the semiclassical Laplacian on  $\tilde{\Omega}$ , and assume that  $A \in L^\infty(\Omega, \mathbb{C}^n)$  and  $q \in L^\infty(\Omega, \mathbb{C})$ . Then for  $0 < h \ll 1$ , we have*

$$h \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)} \leq C \|e^{\varphi/h}(h^2 L_{A,q})e^{-\varphi/h}u\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)}, \quad (2.2)$$

for all  $u \in C_0^\infty(\Omega)$ .

*Proof.* In order to prove the estimate (2.2) it will be convenient to use the following characterization of the semiclassical norm in the Sobolev space  $H^{-1}(\mathbb{R}^n)$ ,

$$\|v\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)} = \sup_{0 \neq \psi \in C_0^\infty(\mathbb{R}^n)} \frac{|\langle v, \psi \rangle_{\mathbb{R}^n}|}{\|\psi\|_{H_{\text{scl}}^1(\mathbb{R}^n)}}, \quad (2.3)$$

where  $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$  is the distribution duality on  $\mathbb{R}^n$ .

Let  $\varphi_\varepsilon = \varphi + \frac{h}{2\varepsilon}\varphi^2$  be the convexified weight with  $\varepsilon > 0$  such that  $0 < h \ll \varepsilon \ll 1$ , and let  $u \in C_0^\infty(\Omega)$ . Then for all  $0 \neq \psi \in C_0^\infty(\mathbb{R}^n)$ , we have

$$\begin{aligned} |\langle e^{\varphi_\varepsilon/h} h^2 A \cdot D(e^{-\varphi_\varepsilon/h} u), \psi \rangle_{\mathbb{R}^n}| &\leq \int_{\mathbb{R}^n} \left| h A \cdot \left( -u \left( 1 + \frac{h}{\varepsilon} \varphi \right) D\varphi + h Du \right) \psi \right| dx \\ &\leq \mathcal{O}(h) \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)} \|\psi\|_{H_{\text{scl}}^1(\mathbb{R}^n)}. \end{aligned}$$

We also obtain that

$$\begin{aligned} |\langle e^{\varphi_\varepsilon/h} h^2 D \cdot (A e^{-\varphi_\varepsilon/h} u), \psi \rangle_{\mathbb{R}^n}| &\leq \int_{\mathbb{R}^n} |h^2 A e^{-\varphi_\varepsilon/h} u \cdot D(e^{\varphi_\varepsilon/h} \psi)| dx \\ &\leq \mathcal{O}(h) \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)} \|\psi\|_{H_{\text{scl}}^1(\mathbb{R}^n)}. \end{aligned}$$

Hence, using (2.3), we get

$$\|e^{\varphi_\varepsilon/h} h^2 A \cdot D(e^{-\varphi_\varepsilon/h} u) + e^{\varphi_\varepsilon/h} h^2 D \cdot (A e^{-\varphi_\varepsilon/h} u)\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)} \leq \mathcal{O}(h) \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)}. \quad (2.4)$$

Notice that the implicit constant in (2.4) only depends on  $\|A\|_{L^\infty(\Omega)}$ ,  $\|\varphi\|_{L^\infty(\Omega)}$  and  $\|D\varphi\|_{L^\infty(\Omega)}$ . Now choosing  $\varepsilon > 0$  sufficiently small but fixed, i.e. independent of  $h$ , we conclude from the estimate (2.1) with  $s = -1$  and the estimate (2.4) that for all  $h > 0$  small enough,

$$\begin{aligned} \|e^{\varphi_\varepsilon/h} (-h^2 \Delta) e^{-\varphi_\varepsilon/h} u + e^{\varphi_\varepsilon/h} h^2 A \cdot D(e^{-\varphi_\varepsilon/h} u) + e^{\varphi_\varepsilon/h} h^2 D \cdot (A e^{-\varphi_\varepsilon/h} u)\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)} \\ \geq \frac{h}{C} \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)}, \quad C > 0. \end{aligned} \quad (2.5)$$

Furthermore, the estimate

$$\|h^2 (A^2 + q) u\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)} \leq \mathcal{O}(h^2) \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)}$$

and the estimate (2.5) imply that for all  $h > 0$  small enough,

$$\|e^{\varphi_\varepsilon/h} (h^2 L_{A,q}) e^{-\varphi_\varepsilon/h} u\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)} \geq \frac{h}{C} \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)}, \quad C > 0.$$

Using that

$$e^{-\varphi_\varepsilon/h} u = e^{-\varphi/h} e^{-\varphi^2/(2\varepsilon)} u,$$

we obtain (2.2). The proof is complete.  $\square$

Let  $\varphi \in C^\infty(\tilde{\Omega}, \mathbb{R})$  be a limiting Carleman weight for  $-h^2 \Delta$  and set  $L_\varphi = e^{\varphi/h} (h^2 L_{A,q}) e^{-\varphi/h}$ . Then we have

$$\langle L_\varphi u, \bar{v} \rangle_\Omega = \langle u, \overline{L_\varphi^* v} \rangle_\Omega, \quad u, v \in C_0^\infty(\Omega),$$

where  $L_\varphi^* = e^{-\varphi/h} (h^2 L_{\bar{A},\bar{q}}) e^{\varphi/h}$  is the formal adjoint of  $L_\varphi$  and  $\langle \cdot, \cdot \rangle_\Omega$  is the distribution duality on  $\Omega$ . We have

$$L_\varphi^* : C_0^\infty(\Omega) \rightarrow H^{-1}(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$$

is bounded, and the estimate (2.2) holds for  $L_\varphi^*$ , since  $-\varphi$  is a limiting Carleman weight as well.

To construct complex geometric optics solutions for the magnetic Schrödinger operator we need to convert the Carleman estimate (2.2) for  $L_\varphi^*$  into the following solvability result. The proof is essentially well-known, and is included here for the convenience of the reader. We shall write

$$\begin{aligned} \|u\|_{H_{\text{scl}}^1(\Omega)}^2 &= \|u\|_{L^2(\Omega)}^2 + \|hDu\|_{L^2(\Omega)}^2, \\ \|v\|_{H_{\text{scl}}^{-1}(\Omega)} &= \sup_{0 \neq \psi \in C_0^\infty(\Omega)} \frac{|\langle v, \psi \rangle_\Omega|}{\|\psi\|_{H_{\text{scl}}^1(\Omega)}}. \end{aligned}$$

**Proposition 2.3.** *Let  $A \in L^\infty(\Omega, \mathbb{C}^n)$ ,  $q \in L^\infty(\Omega, \mathbb{C})$ , and let  $\varphi$  be a limiting Carleman weight for the semiclassical Laplacian on  $\tilde{\Omega}$ . If  $h > 0$  is small enough, then for any  $v \in H^{-1}(\Omega)$ , there is a solution  $u \in H^1(\Omega)$  of the equation*

$$e^{\varphi/h}(h^2 L_{A,q})e^{-\varphi/h}u = v \quad \text{in } \Omega,$$

which satisfies

$$\|u\|_{H_{\text{scl}}^1(\Omega)} \leq \frac{C}{h} \|v\|_{H_{\text{scl}}^{-1}(\Omega)}.$$

*Proof.* Let  $v \in H^{-1}(\Omega)$  and let us consider the following complex linear functional,

$$L : L_\varphi^* C_0^\infty(\Omega) \rightarrow \mathbb{C}, \quad L_\varphi^* w \mapsto \langle w, \bar{v} \rangle_\Omega.$$

By the Carleman estimate (2.2) for  $L_\varphi^*$ , the map  $L$  is well-defined. Let  $w \in C_0^\infty(\Omega)$ . Then we have

$$\begin{aligned} |L(L_\varphi^* w)| &= |\langle w, \bar{v} \rangle_\Omega| \leq \|w\|_{H_{\text{scl}}^1(\mathbb{R}^n)} \|v\|_{H_{\text{scl}}^{-1}(\Omega)} \\ &\leq \frac{C}{h} \|v\|_{H_{\text{scl}}^{-1}(\Omega)} \|L_\varphi^* w\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)}. \end{aligned}$$

By the Hahn-Banach theorem, we may extend  $L$  to a linear continuous functional  $\tilde{L}$  on  $H^{-1}(\mathbb{R}^n)$ , without increasing its norm. By the Riesz representation theorem, there exists  $u \in H^1(\mathbb{R}^n)$  such that for all  $\psi \in H^{-1}(\mathbb{R}^n)$ ,

$$\tilde{L}(\psi) = \langle \psi, \bar{u} \rangle_{\mathbb{R}^n}, \quad \text{and} \quad \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)} \leq \frac{C}{h} \|v\|_{H_{\text{scl}}^{-1}(\Omega)}.$$

Let us now show that  $L_\varphi u = v$  in  $\Omega$ . To that end, let  $w \in C_0^\infty(\Omega)$ . Then

$$\langle L_\varphi u, \bar{w} \rangle_\Omega = \langle u, \overline{L_\varphi^* w} \rangle_{\mathbb{R}^n} = \overline{\tilde{L}(L_\varphi^* w)} = \overline{\langle w, \bar{v} \rangle_\Omega} = \langle v, \bar{w} \rangle_\Omega.$$

The proof is complete.  $\square$

Let  $A \in L^\infty(\Omega, \mathbb{C}^n)$ . We shall extend  $A$  to  $\mathbb{R}^n$  by defining it to be zero in  $\mathbb{R}^n \setminus \Omega$ , and denote this extension by the same letter. Then  $A \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n, \mathbb{C}^n) \subset L^p(\mathbb{R}^n, \mathbb{C}^n)$ ,  $1 \leq p \leq \infty$ .

Let  $\Psi_\tau(x) = \tau^{-n}\Psi(x/\tau)$ ,  $\tau > 0$ , be the usual mollifier with  $\Psi \in C_0^\infty(\mathbb{R}^n)$ ,  $0 \leq \Psi \leq 1$ , and  $\int \Psi dx = 1$ . Then  $A^\sharp = A * \Psi_\tau \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$  and

$$\|A - A^\sharp\|_{L^2(\mathbb{R}^n)} = o(1), \quad \tau \rightarrow 0. \quad (2.6)$$

A direct computation shows that

$$\|\partial^\alpha A^\sharp\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\tau^{-|\alpha|}), \quad \tau \rightarrow 0, \quad \text{for all } \alpha, \quad |\alpha| \geq 0. \quad (2.7)$$

We shall now construct complex geometric optics solutions for the magnetic Schrödinger equation

$$L_{A,q}u = 0 \quad \text{in } \Omega, \quad (2.8)$$

with  $A \in L^\infty(\Omega, \mathbb{C}^n)$  and  $q \in L^\infty(\Omega, \mathbb{C})$ , using the solvability result of Proposition 2.3 and the approximation (2.6). Complex geometric optics solutions are solutions of the form,

$$u(x, \zeta; h) = e^{x \cdot \zeta / h} (a(x, \zeta; h) + r(x, \zeta; h)), \quad (2.9)$$

where  $\zeta \in \mathbb{C}^n$ ,  $\zeta \cdot \zeta = 0$ ,  $|\zeta| \sim 1$ ,  $a$  is a smooth amplitude,  $r$  is a correction term, and  $h > 0$  is a small parameter.

It will be convenient to introduce the following bounded operator,

$$m_A : H^1(\Omega) \rightarrow H^{-1}(\Omega), \quad m_A(u) = D \cdot (Au),$$

where the distribution  $m_A(u)$  is given by

$$\langle m_A(u), v \rangle_\Omega = - \int_\Omega Au \cdot Dv dx, \quad v \in C_0^\infty(\Omega).$$

Let us conjugate  $h^2 L_{A,q}$  by  $e^{x \cdot \zeta / h}$ . First, let us compute  $e^{-x \cdot \zeta / h} \circ h^2 m_A \circ e^{x \cdot \zeta / h}$ . When  $u \in H^1(\Omega)$  and  $v \in C_0^\infty(\Omega)$ , we get

$$\begin{aligned} \langle e^{-x \cdot \zeta / h} h^2 m_A(e^{x \cdot \zeta / h} u), v \rangle_\Omega &= - \int_\Omega h^2 A e^{x \cdot \zeta / h} u \cdot D(e^{-x \cdot \zeta / h} v) dx \\ &= - \int_\Omega (hi\zeta \cdot Auv + h^2 Au \cdot Dv) dx, \end{aligned}$$

and therefore,

$$e^{-x \cdot \zeta / h} \circ h^2 m_A \circ e^{x \cdot \zeta / h} = -hi\zeta \cdot A + h^2 m_A.$$

Furthermore, we obtain that

$$\begin{aligned} e^{-x \cdot \zeta / h} \circ (-h^2 \Delta) \circ e^{x \cdot \zeta / h} &= -h^2 \Delta - 2ih\zeta \cdot D, \\ e^{-x \cdot \zeta / h} \circ h^2(A \cdot D) \circ e^{x \cdot \zeta / h} &= h^2 A \cdot D - hi\zeta \cdot A. \end{aligned}$$

Hence, we have

$$e^{-x \cdot \zeta / h} \circ h^2 L_{A,q} \circ e^{x \cdot \zeta / h} = -h^2 \Delta - 2ih\zeta \cdot D + h^2 A \cdot D - 2hi\zeta \cdot A + h^2 m_A + h^2(A^2 + q). \quad (2.10)$$

We shall consider  $\zeta$  depending slightly on  $h$ , i.e.  $\zeta = \zeta_0 + \zeta_1$  with  $\zeta_0$  being independent of  $h$  and  $\zeta_1 = \mathcal{O}(h)$  as  $h \rightarrow 0$ . We also assume that  $|\operatorname{Re} \zeta_0| = |\operatorname{Im} \zeta_0| = 1$ . Then we write (2.10) as follows,

$$\begin{aligned} e^{-x \cdot \zeta/h} \circ h^2 L_{A,q} \circ e^{x \cdot \zeta/h} = & -h^2 \Delta - 2ih\zeta_0 \cdot D - 2ih\zeta_1 \cdot D + h^2 A \cdot D - 2hi\zeta_0 \cdot A^\sharp \\ & - 2hi\zeta_0 \cdot (A - A^\sharp) - 2hi\zeta_1 \cdot A + h^2 m_A + h^2(A^2 + q). \end{aligned}$$

In order that (2.9) be a solution of (2.8), we require that

$$\zeta_0 \cdot Da + \zeta_0 \cdot A^\sharp a = 0 \quad \text{in } \mathbb{R}^n, \quad (2.11)$$

and

$$\begin{aligned} e^{-x \cdot \zeta/h} h^2 L_{A,q} e^{x \cdot \zeta/h} r = & -(-h^2 \Delta a + h^2 A \cdot Da + h^2 m_A(a) + h^2(A^2 + q)a) \\ & + 2ih\zeta_1 \cdot Da + 2hi\zeta_0 \cdot (A - A^\sharp)a + 2hi\zeta_1 \cdot Aa =: g \quad \text{in } \Omega. \end{aligned} \quad (2.12)$$

The equation (2.11) is the first transport equation and one looks for its solution in the form  $a = e^{\Phi^\sharp}$ , where  $\Phi^\sharp$  solves the equation

$$\zeta_0 \cdot \nabla \Phi^\sharp + i\zeta_0 \cdot A^\sharp = 0 \quad \text{in } \mathbb{R}^n. \quad (2.13)$$

As  $\zeta_0 \cdot \zeta_0 = 0$  and  $|\operatorname{Re} \zeta_0| = |\operatorname{Im} \zeta_0| = 1$ , the operator  $N_{\zeta_0} := \zeta_0 \cdot \nabla$  is the  $\bar{\partial}$ -operator in suitable linear coordinates. Let us introduce an inverse operator defined by

$$(N_{\zeta_0}^{-1} f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{f(x - y_1 \operatorname{Re} \zeta_0 - y_2 \operatorname{Im} \zeta_0)}{y_1 + iy_2} dy_1 dy_2, \quad f \in C_0(\mathbb{R}^n).$$

We have the following result, see [17, Lemma 4.6].

**Lemma 2.4.** *Let  $f \in W^{k,\infty}(\mathbb{R}^n)$ ,  $k \geq 0$ , with  $\operatorname{supp}(f) \subset B(0, R)$ . Then  $\Phi = N_{\zeta_0}^{-1} f \in W^{k,\infty}(\mathbb{R}^n)$  satisfies  $N_{\zeta_0} \Phi = f$  in  $\mathbb{R}^n$ , and we have*

$$\|\Phi\|_{W^{k,\infty}(\mathbb{R}^n)} \leq C \|f\|_{W^{k,\infty}(\mathbb{R}^n)}, \quad (2.14)$$

where  $C = C(R)$ . If  $f \in C_0(\mathbb{R}^n)$ , then  $\Phi \in C(\mathbb{R}^n)$ .

Thanks to Lemma 2.4, the function  $\Phi^\sharp(x, \zeta_0; \tau) := N_{\zeta_0}^{-1}(-i\zeta_0 \cdot A^\sharp) \in C^\infty(\mathbb{R}^n)$  satisfies the equation (2.13). Furthermore, the estimates (2.7) and (2.14) imply that

$$\|\partial^\alpha \Phi^\sharp\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha \tau^{-|\alpha|}, \quad \text{for all } \alpha, \quad |\alpha| \geq 0. \quad (2.15)$$

Owing to [21, Lemma 3.1], we have the following result, where we use the norms

$$\|f\|_{L_\delta^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (1 + |x|^2)^\delta |f(x)|^2 dx.$$

**Lemma 2.5.** *Let  $-1 < \delta < 0$  and let  $f \in L_{\delta+1}^2(\mathbb{R}^n)$ . Then there exists a constant  $C > 0$ , independent of  $\zeta_0$ , such that*

$$\|N_{\zeta_0}^{-1} f\|_{L_\delta^2(\mathbb{R}^n)} \leq C \|f\|_{L_{\delta+1}^2(\mathbb{R}^n)}.$$



Setting  $\Phi(\cdot, \zeta_0) := N_{\zeta_0}^{-1}(-i\zeta_0 \cdot A) \in L^\infty(\mathbb{R}^n)$ , it follows from Lemma 2.5 and the estimate (2.6) that  $\Phi^\sharp(\cdot, \zeta_0; \tau)$  converges to  $\Phi(\cdot, \zeta_0)$  in  $L_{\text{loc}}^2(\mathbb{R}^n)$  as  $\tau \rightarrow 0$ .

Let us turn now to the equation (2.12). First notice that the right hand side  $g$  of (2.12) belongs to  $H^{-1}(\Omega)$  and we would like to estimate  $\|g\|_{H_{\text{scl}}^{-1}(\Omega)}$ . To that end, let  $0 \neq \psi \in C_0^\infty(\Omega)$ . Then using (2.15) and the fact that  $\zeta_1 = \mathcal{O}(h)$ , we get by the Cauchy–Schwarz inequality,

$$\begin{aligned} |\langle h^2 \Delta a, \psi \rangle_\Omega| &\leq \mathcal{O}(h^2/\tau^2) \|\psi\|_{L^2(\Omega)} \leq \mathcal{O}(h^2/\tau^2) \|\psi\|_{H_{\text{scl}}^1(\Omega)}, \\ |\langle h^2 A \cdot Da, \psi \rangle_\Omega| &\leq \mathcal{O}(h^2/\tau) \|\psi\|_{H_{\text{scl}}^1(\Omega)}, \\ |\langle 2ih\zeta_1 \cdot Da, \psi \rangle_\Omega| &\leq \mathcal{O}(h^2/\tau) \|\psi\|_{H_{\text{scl}}^1(\Omega)}, \\ |\langle 2hi\zeta_1 \cdot Aa, \psi \rangle_\Omega| &\leq \mathcal{O}(h^2) \|\psi\|_{H_{\text{scl}}^1(\Omega)}. \end{aligned}$$

Using (2.6) and (2.15), we have

$$\begin{aligned} |\langle 2hi\zeta_0 \cdot (A - A^\sharp)a, \psi \rangle_\Omega| &\leq \mathcal{O}(h) \|a\|_{L^\infty(\mathbb{R}^n)} \|A - A^\sharp\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \\ &\leq \mathcal{O}(h) o_{\tau \rightarrow 0}(1) \|\psi\|_{H_{\text{scl}}^1(\Omega)}. \end{aligned}$$

With the help of (2.6), (2.7), and (2.15), we obtain that

$$\begin{aligned} |\langle h^2 m_A(a), \psi \rangle_\Omega| &\leq \left| \int_\Omega h^2 A^\sharp a \cdot D\psi dx \right| + \left| \int_\Omega h^2 (A - A^\sharp)a \cdot D\psi dx \right| \\ &\leq \left| \int_\Omega h^2 (D \cdot (A^\sharp a)) \psi dx \right| + \mathcal{O}(h) \|A - A^\sharp\|_{L^2(\Omega)} \|hD\psi\|_{L^2(\Omega)} \\ &\leq (\mathcal{O}(h^2/\tau) + \mathcal{O}(h) o_{\tau \rightarrow 0}(1)) \|\psi\|_{H_{\text{scl}}^1(\Omega)}. \end{aligned}$$

We also have  $\|h^2(A^2 + q)a\|_{L^2(\Omega)} \leq \mathcal{O}(h^2)$ . Thus, from the above estimates, we conclude that

$$\|g\|_{H_{\text{scl}}^{-1}(\Omega)} \leq \mathcal{O}(h^2/\tau^2) + \mathcal{O}(h) o_{\tau \rightarrow 0}(1).$$

Choosing now  $\tau = h^\sigma$  with some  $\sigma, 0 < \sigma < 1/2$ , we get

$$\|g\|_{H_{\text{scl}}^{-1}(\Omega)} = o(h) \quad \text{as } h \rightarrow 0. \quad (2.16)$$

Thanks to Proposition 2.3 and (2.16), for  $h > 0$  small enough, there exists a solution  $r \in H^1(\Omega)$  of (2.12) such that  $\|r\|_{H_{\text{scl}}^1(\Omega)} = o(1)$  as  $h \rightarrow 0$ .

The discussion led in this section can be summarized in the following proposition.

**Proposition 2.6.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open set. Let  $A \in L^\infty(\Omega, \mathbb{C}^n)$ ,  $q \in L^\infty(\Omega, \mathbb{C})$ , and let  $\zeta \in \mathbb{C}^n$  be such that  $\zeta \cdot \zeta = 0$ ,  $\zeta = \zeta_0 + \zeta_1$  with  $\zeta_0$  being independent of  $h > 0$ ,  $|\text{Re } \zeta_0| = |\text{Im } \zeta_0| = 1$ , and  $\zeta_1 = \mathcal{O}(h)$  as  $h \rightarrow 0$ . Then for all  $h > 0$  small enough, there exists a solution  $u(x, \zeta; h) \in H^1(\Omega)$  to the magnetic Schrödinger equation  $L_{A,q}u = 0$  in  $\Omega$ , of the form*

$$u(x, \zeta; h) = e^{x \cdot \zeta / h} (e^{\Phi^\sharp(x, \zeta_0; h)} + r(x, \zeta; h)).$$

The function  $\Phi^\sharp(\cdot, \zeta_0; h) \in C^\infty(\mathbb{R}^n)$  satisfies  $\|\partial^\alpha \Phi^\sharp\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha h^{-\sigma|\alpha|}$ ,  $0 < \sigma < 1/2$ , for all  $\alpha$ ,  $|\alpha| \geq 0$ , and  $\Phi^\sharp(\cdot, \zeta_0; h)$  converges to  $\Phi(\cdot, \zeta_0) := N_{\zeta_0}^{-1}(-i\zeta_0 \cdot A) \in L^\infty(\mathbb{R}^n)$  in  $L^2_{\text{loc}}(\mathbb{R}^n)$  as  $h \rightarrow 0$ . Here we have extended  $A$  by zero to  $\mathbb{R}^n \setminus \Omega$ . The remainder  $r$  is such that  $\|r\|_{H^1_{\text{scl}}(\Omega)} = o(1)$  as  $h \rightarrow 0$ .

### 3. PROOF OF THEOREM 1.1

Let us begin by recalling the following auxiliary, essentially well-known, result which shows that the set of the Cauchy data for the magnetic Schrödinger operator remains unchanged if the gradient of a function, vanishing along the boundary, is added to the magnetic potential, see [17, Lemma 4.1], [20].

**Lemma 3.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set, let  $A \in L^\infty(\Omega, \mathbb{C}^n)$ ,  $q \in L^\infty(\Omega, \mathbb{C})$ , and let  $\psi \in W^{1,\infty}$  in a neighborhood of  $\bar{\Omega}$ . Then we have*

$$e^{-i\psi} \circ L_{A,q} \circ e^{i\psi} = L_{A+\nabla\psi,q}. \quad (3.1)$$

If furthermore,  $\psi|_{\partial\Omega} = 0$  then

$$C_{A,q} = C_{A+\nabla\psi,q}. \quad (3.2)$$

*Proof.* Let us notice first that the assumption that  $\psi \in W^{1,\infty}$  in a neighborhood of  $\bar{\Omega}$  implies that  $\psi$  is Lipschitz continuous on  $\bar{\Omega}$ , so that  $\psi|_{\partial\Omega}$  is well-defined pointwise.

Since (3.1) follows by a direct computation, only (3.2) has to be established. To that end, let  $u \in H^1(\Omega)$  be a solution to  $L_{A,q}u = 0$  in  $\Omega$ . Then  $e^{-i\psi}u \in H^1(\Omega)$  satisfies  $L_{A+\nabla\psi,q}(e^{-i\psi}u) = 0$  in  $\Omega$ . Let us show that  $T(e^{-i\psi}u) = Tu$ . In other words, we have to check that

$$u(e^{-i\psi} - 1) \in H_0^1(\Omega). \quad (3.3)$$

Since the function  $e^{-i\psi} - 1$  is Lipschitz continuous on  $\bar{\Omega}$  and vanishes along  $\partial\Omega$ , we have  $|e^{-i\psi(x)} - 1| \leq Cd(x)$  for any  $x \in \Omega$  and some constant  $C > 0$ . Here  $d(x)$  is the distance from  $x$  to the boundary of  $\Omega$ . Then (3.3) follows from the following fact: if  $v \in H^1(\Omega)$  and  $v/d \in L^2(\Omega)$ , then  $v \in H_0^1(\Omega)$ , see [6, Theorem 3.4, p. 223].

Let us now show that  $N_{A+\nabla\psi,q}(e^{-i\psi}u) = N_{A,q}u$ . To that end, first as above, one observes that for  $g \in H^1(\Omega)$ , we have  $[g] = [e^{i\psi}g]$ . Thus,

$$(N_{A+\nabla\psi,q}(e^{-i\psi}u), [g])_\Omega = (N_{A+\nabla\psi,q}(e^{-i\psi}u), [e^{i\psi}g])_\Omega = (N_{A,q}(u), [g])_\Omega,$$

for any  $[g] \in H^1(\Omega)/H_0^1(\Omega)$ , and therefore,  $C_{A,q} \subset C_{A+\nabla\psi,q}$ . The proof is complete.  $\square$

The first step in the proof of Theorem 1.1 is the derivation of the following integral identity based on the fact that  $C_{A_1,q_1} = C_{A_2,q_2}$ , see also [17, Lemma 4.3].

**Proposition 3.2.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open set. Assume that  $A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n)$  and  $q_1, q_2 \in L^\infty(\Omega, \mathbb{C})$ . If  $C_{A_1, q_1} = C_{A_2, q_2}$ , then the following integral identity*

$$\int_{\Omega} i(A_1 - A_2) \cdot (u_1 \nabla \overline{u_2} - \overline{u_2} \nabla u_1) dx + \int_{\Omega} (A_1^2 - A_2^2 + q_1 - q_2) u_1 \overline{u_2} dx = 0 \quad (3.4)$$

*holds for any  $u_1, u_2 \in H^1(\Omega)$  satisfying  $L_{A_1, q_1} u_1 = 0$  in  $\Omega$  and  $L_{\overline{A_2}, \overline{q_2}} u_2 = 0$  in  $\Omega$ , respectively.*

*Proof.* Let  $u_1, u_2 \in H^1(\Omega)$  be solutions to  $L_{A_1, q_1} u_1 = 0$  in  $\Omega$  and  $L_{\overline{A_2}, \overline{q_2}} u_2 = 0$  in  $\Omega$ , respectively. Then the fact that  $C_{A_1, q_1} = C_{A_2, q_2}$  implies that there is  $v_2 \in H^1(\Omega)$  satisfying  $L_{A_2, q_2} v_2 = 0$  in  $\Omega$  such that

$$Tu_1 = Tv_2 \quad \text{and} \quad N_{A_1, q_1} u_1 = N_{A_2, q_2} v_2.$$

This together with (1.2) shows that

$$(N_{A_1, q_1} u_1, [\overline{u_2}])_{\Omega} = (N_{A_2, q_2} v_2, [\overline{u_2}])_{\Omega} = \overline{(N_{\overline{A_2}, \overline{q_2}} u_2, [\overline{v_2}])_{\Omega}} = \overline{(N_{\overline{A_2}, \overline{q_2}} u_2, [\overline{u_1}])_{\Omega}}.$$

Then the integral identity (3.4) follows from the definition (1.2) of  $N_{A_1, q_1} u_1$  and  $N_{\overline{A_2}, \overline{q_2}} u_2$ . The proof is complete.  $\square$

We shall use the integral identity (3.4) with  $u_1$  and  $u_2$  being complex geometric optics solutions for the magnetic Schrödinger equations in  $\Omega$ . To construct such solutions, let  $\xi, \mu_1, \mu_2 \in \mathbb{R}^n$  be such that  $|\mu_1| = |\mu_2| = 1$  and  $\mu_1 \cdot \mu_2 = \mu_1 \cdot \xi = \mu_2 \cdot \xi = 0$ . Similarly to [20], we set

$$\zeta_1 = \frac{ih\xi}{2} + \mu_1 + i\sqrt{1 - h^2 \frac{|\xi|^2}{4}} \mu_2, \quad \zeta_2 = -\frac{ih\xi}{2} - \mu_1 + i\sqrt{1 - h^2 \frac{|\xi|^2}{4}} \mu_2, \quad (3.5)$$

so that  $\zeta_j \cdot \zeta_j = 0$ ,  $j = 1, 2$ , and  $(\zeta_1 + \overline{\zeta_2})/h = i\xi$ . Here  $h > 0$  is a small enough semiclassical parameter. Moreover,  $\zeta_1 = \mu_1 + i\mu_2 + \mathcal{O}(h)$  and  $\zeta_2 = -\mu_1 + i\mu_2 + \mathcal{O}(h)$  as  $h \rightarrow 0$ .

By Proposition 2.6, for all  $h > 0$  small enough, there exists a solution  $u_1(x, \zeta_1; h) \in H^1(\Omega)$  to the magnetic Schrödinger equation  $L_{A_1, q_1} u_1 = 0$  in  $\Omega$ , of the form

$$u_1(x, \zeta_1; h) = e^{x \cdot \zeta_1 / h} (e^{\Phi_1^\sharp(x, \mu_1 + i\mu_2; h)} + r_1(x, \zeta_1; h)), \quad (3.6)$$

where  $\Phi_1^\sharp(\cdot, \mu_1 + i\mu_2; h) \in C^\infty(\mathbb{R}^n)$  satisfies the estimate

$$\|\partial^\alpha \Phi_1^\sharp\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha h^{-\sigma|\alpha|}, \quad 0 < \sigma < 1/2, \quad (3.7)$$

for all  $\alpha$ ,  $|\alpha| \geq 0$ ,  $\Phi_1^\sharp(\cdot, \mu_1 + i\mu_2; h)$  converges to

$$\Phi_1(\cdot, \mu_1 + i\mu_2) := N_{\mu_1 + i\mu_2}^{-1}(-i(\mu_1 + i\mu_2) \cdot A_1) \in L^\infty(\mathbb{R}^n) \quad (3.8)$$

in  $L_{\text{loc}}^2(\mathbb{R}^n)$  as  $h \rightarrow 0$ , and

$$\|r_1\|_{H_{\text{scl}}^1(\Omega)} = o(1) \quad \text{as} \quad h \rightarrow 0. \quad (3.9)$$

Similarly, for all  $h > 0$  small enough, there exists a solution  $u_2(x, \zeta_2; h) \in H^1(\Omega)$  to the magnetic Schrödinger equation  $L_{\overline{A_2}, \overline{q_2}} u_2 = 0$  in  $\Omega$ , of the form

$$u_2(x, \zeta_2; h) = e^{x \cdot \zeta_2 / h} (e^{\Phi_2^\sharp(x, -\mu_1 + i\mu_2; h)} + r_2(x, \zeta_2; h)), \quad (3.10)$$

where  $\Phi_2^\sharp(\cdot, -\mu_1 + i\mu_2; h) \in C^\infty(\mathbb{R}^n)$  satisfies the estimate

$$\|\partial^\alpha \Phi_2^\sharp\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha h^{-\sigma|\alpha|}, \quad 0 < \sigma < 1/2, \quad (3.11)$$

for all  $\alpha$ ,  $|\alpha| \geq 0$ . Furthermore,  $\Phi_2^\sharp(\cdot, -\mu_1 + i\mu_2; h)$  converges to

$$\Phi_2(\cdot, -\mu_1 + i\mu_2) := N_{-\mu_1 + i\mu_2}^{-1}(-i(-\mu_1 + i\mu_2) \cdot \overline{A_2}) \in L^\infty(\mathbb{R}^n) \quad (3.12)$$

in  $L_{\text{loc}}^2(\mathbb{R}^n)$  as  $h \rightarrow 0$ , and

$$\|r_2\|_{H_{\text{scl}}^1(\Omega)} = o(1) \quad \text{as } h \rightarrow 0. \quad (3.13)$$

We shall next substitute  $u_1$  and  $u_2$ , given by (3.6) and (3.10), into the integral identity (3.4), multiply it by  $h$ , and let  $h \rightarrow 0$ . We first compute

$$\begin{aligned} hu_1 \nabla \overline{u_2} &= \overline{\zeta_2} e^{ix \cdot \xi} (e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} + e^{\Phi_1^\sharp} \overline{r_2} + r_1 e^{\overline{\Phi_2^\sharp}} + r_1 \overline{r_2}) \\ &\quad + h e^{ix \cdot \xi} (e^{\Phi_1^\sharp} \nabla e^{\overline{\Phi_2^\sharp}} + e^{\Phi_1^\sharp} \nabla \overline{r_2} + r_1 \nabla e^{\overline{\Phi_2^\sharp}} + r_1 \nabla \overline{r_2}). \end{aligned}$$

Recall that  $\overline{\zeta_2} = -\mu_1 - i\mu_2 + \mathcal{O}(h)$ . We shall show that

$$(\mu_1 + i\mu_2) \cdot \int_{\Omega} (A_1 - A_2) e^{ix \cdot \xi} e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} dx \rightarrow (\mu_1 + i\mu_2) \cdot \int_{\Omega} (A_1 - A_2) e^{ix \cdot \xi} e^{\Phi_1 + \overline{\Phi_2}} dx,$$

as  $h \rightarrow 0$ , where  $\Phi_1$  and  $\Phi_2$  are defined by (3.8) and (3.12), respectively. To that end, we have

$$\begin{aligned} \left| (\mu_1 + i\mu_2) \cdot \int_{\Omega} (A_1 - A_2) e^{ix \cdot \xi} (e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} - e^{\Phi_1 + \overline{\Phi_2}}) dx \right| &\leq C \|e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} - e^{\Phi_1 + \overline{\Phi_2}}\|_{L^2(\Omega)} \\ &\leq C \|\Phi_1^\sharp + \overline{\Phi_2^\sharp} - \Phi_1 - \overline{\Phi_2}\|_{L^2(\Omega)} \rightarrow 0, \end{aligned}$$

as  $h \rightarrow 0$ . Here we have used the inequality

$$|e^z - e^w| \leq |z - w| e^{\max(\operatorname{Re} z, \operatorname{Re} w)}, \quad z, w \in \mathbb{C}, \quad (3.14)$$

obtained by integration of  $e^z$  from  $z$  to  $w$ , and the fact that  $\Phi_j, \Phi_j^\sharp \in L^\infty(\mathbb{R}^n)$ ,  $j = 1, 2$ , and  $\|\Phi_j^\sharp\|_{L^\infty(\mathbb{R}^n)} \leq C$  uniformly in  $h$ .

Now using the estimates (3.7), (3.9), (3.11) and (3.13), we get

$$\begin{aligned} &\left| \int_{\Omega} i(A_1 - A_2) \cdot \overline{\zeta_2} e^{ix \cdot \xi} (e^{\Phi_1^\sharp} \overline{r_2} + r_1 e^{\overline{\Phi_2^\sharp}} + r_1 \overline{r_2}) dx \right| \\ &\leq C \|A_1 - A_2\|_{L^\infty} (\|e^{\Phi_1^\sharp}\|_{L^2} \|\overline{r_2}\|_{L^2} + \|r_1\|_{L^2} \|e^{\overline{\Phi_2^\sharp}}\|_{L^2} + \|r_1\|_{L^2} \|\overline{r_2}\|_{L^2}) = o(1), \end{aligned}$$

as  $h \rightarrow 0$ . We also obtain that

$$\left| \int_{\Omega} hi(A_1 - A_2) \cdot e^{ix \cdot \xi} (e^{\Phi_1^\sharp} \nabla e^{\overline{\Phi_2^\sharp}} + e^{\Phi_1^\sharp} \nabla \overline{r_2} + r_1 \nabla e^{\overline{\Phi_2^\sharp}} + r_1 \nabla \overline{r_2}) dx \right| \leq \mathcal{O}(h)(h^{-\sigma} + h^{-1}o(1) + o(1)h^{-\sigma} + o(1)h^{-1}) = o(1),$$

as  $h \rightarrow 0$ . Here  $0 < \sigma < 1/2$ . Furthermore,

$$\left| h \int_{\Omega} (A_1^2 - A_2^2 + q_1 - q_2) e^{ix \cdot \xi} (e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} + e^{\Phi_1^\sharp} \overline{r_2} + r_1 e^{\overline{\Phi_2^\sharp}} + r_1 \overline{r_2}) dx \right| = \mathcal{O}(h),$$

as  $h \rightarrow 0$ . Hence, substituting  $u_1$  and  $u_2$ , given by (3.6) and (3.10), into the integral identity (3.4), multiplying it by  $h$ , and letting  $h \rightarrow 0$ , we get

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1 - A_2) e^{ix \cdot \xi} e^{\Phi_1(x, \mu_1 + i\mu_2) + \overline{\Phi_2(x, -\mu_1 + i\mu_2)}} dx = 0, \quad (3.15)$$

where

$$\begin{aligned} \Phi_1 &= N_{\mu_1 + i\mu_2}^{-1}(-i(\mu_1 + i\mu_2) \cdot A_1) \in L^\infty(\mathbb{R}^n), \\ \Phi_2 &= N_{-\mu_1 + i\mu_2}^{-1}(-i(-\mu_1 + i\mu_2) \cdot \overline{A_2}) \in L^\infty(\mathbb{R}^n). \end{aligned}$$

Notice that the integration in (3.15) is extended to all of  $\mathbb{R}^n$ , since  $A_1 = A_2 = 0$  on  $\mathbb{R}^n \setminus \Omega$ .

The next step is to remove the function  $e^{\Phi_1 + \overline{\Phi_2}}$  in the integral (3.15). First using the following properties of the Cauchy transform,

$$\overline{N_\zeta^{-1}f} = N_{\overline{\zeta}}^{-1}\overline{f}, \quad N_{-\zeta}^{-1}f = -N_\zeta^{-1}f,$$

we see that

$$\Phi_1 + \overline{\Phi_2} = N_{\mu_1 + i\mu_2}^{-1}(-i(\mu_1 + i\mu_2) \cdot (A_1 - A_2)). \quad (3.16)$$

We have the following result.

**Proposition 3.3.** *Let  $\xi, \mu_1, \mu_2 \in \mathbb{R}^n$ ,  $n \geq 3$ , be such that  $|\mu_1| = |\mu_2| = 1$  and  $\mu_1 \cdot \mu_2 = \mu_1 \cdot \xi = \mu_2 \cdot \xi = 0$ . Let  $W \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n, \mathbb{C}^n)$  and  $\phi = N_{\mu_1 + i\mu_2}^{-1}(-i(\mu_1 + i\mu_2) \cdot W)$ . Then*

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} W(x) e^{ix \cdot \xi} e^{\phi(x)} dx = (\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} W(x) e^{ix \cdot \xi} dx. \quad (3.17)$$

*Proof.* The statement of the proposition for  $W \in C_0(\mathbb{R}^n, \mathbb{C}^n)$  is due to [7], with similar ideas appearing in [20]. See also [18, Lemma 6.2]. For the completeness and convenience of the reader, we shall give a complete proof of the proposition here.

Assume first that  $W \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$ . Then by Lemma 2.4 we have

$$\phi = N_{\mu_1 + i\mu_2}^{-1}(-i(\mu_1 + i\mu_2) \cdot W) \in C^\infty(\mathbb{R}^n). \quad (3.18)$$

We can always assume that  $\mu_1 = (1, 0, \dots, 0)$  and  $\mu_2 = (0, 1, 0, \dots, 0)$ , so that  $\xi = (0, 0, \xi'')$ ,  $\xi'' \in \mathbb{R}^{n-2}$ , and therefore,

$$(\partial_{x_1} + i\partial_{x_2})\phi = -i(\mu_1 + i\mu_2) \cdot W \quad \text{in } \mathbb{R}^n.$$

Hence, writing  $x = (x', x'')$ ,  $x' = (x_1, x_2)$ ,  $x'' \in \mathbb{R}^{n-2}$ , we get

$$\begin{aligned} (\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} W(x) e^{ix \cdot \xi} e^{\phi(x)} dx &= i \int_{\mathbb{R}^n} e^{ix'' \cdot \xi''} e^{\phi(x)} (\partial_{x_1} + i\partial_{x_2})\phi(x) dx \\ &= i \int_{\mathbb{R}^{n-2}} e^{ix'' \cdot \xi''} h(x'') dx'', \end{aligned}$$

where

$$\begin{aligned} h(x'') &= \int_{\mathbb{R}^2} (\partial_{x_1} + i\partial_{x_2}) e^{\phi(x)} dx' = \lim_{R \rightarrow \infty} \int_{|x'| \leq R} (\partial_{x_1} + i\partial_{x_2}) e^{\phi(x)} dx' \\ &= \lim_{R \rightarrow \infty} \int_{|x'|=R} e^{\phi(x)} (\nu_1 + i\nu_2) dS_R(x'). \end{aligned}$$

Here  $\nu = (\nu_1, \nu_2)$  is the unit outer normal to the circle  $|x'| = R$ , and we have used the Gauss theorem.

It follows from (3.18) that  $|\phi(x', x'')| = \mathcal{O}(1/|x'|)$  as  $|x'| \rightarrow \infty$ . Hence, we have

$$e^\phi = 1 + \phi + \mathcal{O}(|\phi|^2) = 1 + \phi + \mathcal{O}(|x'|^{-2}) \quad \text{as } |x'| \rightarrow \infty.$$

Since

$$\begin{aligned} \int_{|x'|=R} (\nu_1 + i\nu_2) dS_R(x') &= \int_{|x'| \leq R} (\partial_{x_1} + i\partial_{x_2})(1) dx' = 0, \\ \left| \int_{|x'|=R} \mathcal{O}(|x'|^{-2}) (\nu_1 + i\nu_2) dS_R(x') \right| &\leq \mathcal{O}(R^{-1}) \quad \text{as } R \rightarrow \infty, \end{aligned}$$

we obtain that

$$\begin{aligned} h(x'') &= \lim_{R \rightarrow \infty} \int_{|x'|=R} \phi(x) (\nu_1 + i\nu_2) dS_R(x') = \lim_{R \rightarrow \infty} \int_{|x'| \leq R} (\partial_{x_1} + i\partial_{x_2})\phi(x) dx' \\ &= - \int_{\mathbb{R}^2} i(\mu_1 + i\mu_2) \cdot W(x) dx', \end{aligned}$$

which shows (3.17) for  $W \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$ .

To prove (3.17) for  $W \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n, \mathbb{C}^n)$ , consider the regularizations  $W_j = \chi_j * W \in C_0^\infty(\mathbb{R}^n)$ . Here  $\chi_j(x) = j^n \chi(jx)$  is the usual mollifier with  $0 \leq \chi \in C_0^\infty(\mathbb{R}^n)$  such that  $\int \chi dx = 1$ . Then  $W_j \rightarrow W$  in  $L^2(\mathbb{R}^n)$  as  $j \rightarrow \infty$  and

$$\|W_j\|_{L^\infty(\mathbb{R}^n)} \leq \|W\|_{L^\infty(\mathbb{R}^n)} \|\chi_j\|_{L^1(\mathbb{R}^n)} = \|W\|_{L^\infty(\mathbb{R}^n)}, \quad j = 1, 2, \dots \quad (3.19)$$

Furthermore, there is a compact set  $K \subset \subset \mathbb{R}^n$  such that  $\text{supp}(W_j), \text{supp}(W) \subset K$ ,  $j = 1, 2, \dots$

We set  $\phi_j = N_{\mu_1+i\mu_2}^{-1}(-i(\mu_1+i\mu_2) \cdot W_j) \in C^\infty(\mathbb{R}^n)$ . Then by Lemma 2.5, we know that  $\phi_j \rightarrow \phi$  in  $L_{\text{loc}}^2(\mathbb{R}^n)$  as  $j \rightarrow \infty$ . Lemma 2.4 together with the estimate (3.19) implies that

$$\|\phi_j\|_{L^\infty(\mathbb{R}^n)} \leq C\|W_j\|_{L^\infty(\mathbb{R}^n)} \leq C\|W\|_{L^\infty(\mathbb{R}^n)}, \quad j = 1, 2, \dots \quad (3.20)$$

For  $j = 1, 2, \dots$ , we have

$$(\mu_1 + i\mu_2) \cdot \int_K W_j(x) e^{ix \cdot \xi} e^{\phi_j(x)} dx = (\mu_1 + i\mu_2) \cdot \int_K W_j(x) e^{ix \cdot \xi} dx. \quad (3.21)$$

The fact that the integral in right hand side of (3.21) converges to the integral in the right hand side of (3.17) as  $j \rightarrow \infty$  follows from the estimate

$$\left| (\mu_1 + i\mu_2) \cdot \int_K (W_j(x) - W(x)) e^{ix \cdot \xi} dx \right| \leq C\|W_j - W\|_{L^2(K)} \rightarrow 0, \quad j \rightarrow \infty.$$

In order to show that the integral in the left hand side of (3.21) converges to the integral in the left hand side of (3.17) as  $j \rightarrow \infty$ , we establish that  $I_1 + I_2 \rightarrow 0$  as  $j \rightarrow \infty$ , where

$$\begin{aligned} I_1 &:= (\mu_1 + i\mu_2) \cdot \int_K (W_j(x) - W(x)) e^{ix \cdot \xi} e^{\phi_j(x)} dx, \\ I_2 &:= (\mu_1 + i\mu_2) \cdot \int_K W(x) e^{ix \cdot \xi} (e^{\phi_j(x)} - e^{\phi(x)}) dx. \end{aligned}$$

Using (3.20), we have

$$|I_1| \leq C e^{\|\phi_j\|_{L^\infty(\mathbb{R}^n)}} \int_K |W_j(x) - W(x)| dx \leq C\|W_j - W\|_{L^2(K)} \rightarrow 0, \quad j \rightarrow \infty.$$

Using (3.14) and (3.20), we get

$$|I_2| \leq C\|W\|_{L^\infty(\mathbb{R}^n)} \|e^{\phi_j(x)} - e^{\phi(x)}\|_{L^2(K)} \leq C\|\phi_j - \phi\|_{L^2(K)} \rightarrow 0, \quad j \rightarrow \infty.$$

Here we have also used that  $\phi_j \rightarrow \phi$  in  $L_{\text{loc}}^2(\mathbb{R}^n)$  as  $j \rightarrow \infty$ . Hence, passing to the limit as  $j \rightarrow \infty$  in (3.21), we obtain the identity (3.17). The proof is complete.  $\square$

By Proposition 3.3 we conclude from (3.15) and (3.16) that

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1(x) - A_2(x)) e^{ix \cdot \xi} dx = 0. \quad (3.22)$$

It follows from (3.22) that  $\mu \cdot (\widehat{A}_1(\xi) - \widehat{A}_2(\xi)) = 0$  whenever  $\mu, \xi \in \mathbb{R}^n$  are such that  $\mu \cdot \xi = 0$ . Here  $\widehat{A}_j$  is the Fourier transform of  $A_j$ ,  $j = 1, 2$ . Let  $\mu_{jk}(\xi) = \xi_j e_k - \xi_k e_j$  for  $j \neq k$ , where  $e_1, \dots, e_n$  is the standard basis of  $\mathbb{R}^n$ . Then  $\mu_{jk}(\xi) \cdot \xi = 0$ , and therefore,

$$\xi_j (\widehat{A}_{1,k}(\xi) - \widehat{A}_{2,k}(\xi)) - \xi_k (\widehat{A}_{1,j}(\xi) - \widehat{A}_{2,j}(\xi)) = 0.$$

Hence,  $\partial_{x_j}(A_{1,k} - A_{2,k}) - \partial_{x_k}(A_{1,j} - A_{2,j}) = 0$  in  $\mathbb{R}^n$  in the sense of distributions, for  $j \neq k$ , and thus,  $d(A_1 - A_2) = 0$  in  $\mathbb{R}^n$ .

Our next goal is to show that  $q_1 = q_2$  in  $\Omega$ . First, viewing  $A_1 - A_2$  as a 1-current and using the Poincaré lemma for currents, we conclude that there is  $\psi \in \mathcal{D}'(\mathbb{R}^n)$  such that  $d\psi = A_1 - A_2 \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n)$  in  $\mathbb{R}^n$ , see [16]. It follows from [10, Theorem 4.5.11] that  $\psi$  is continuous on  $\mathbb{R}^n$ , and since  $\psi$  is constant near infinity, we have  $\psi \in L^\infty(\mathbb{R}^n)$ . Therefore,  $\psi \in W^{1,\infty}(\mathbb{R}^n)$ , and without loss of generality, we may assume that there is an open ball  $B$  such that  $\Omega \subset\subset B$  and  $\text{supp}(\psi) \subset B$ .

We want to add  $\nabla\psi$  to the potential  $A_2$  without changing the set of the Cauchy data for  $L_{A_2,q_2}$  on the ball  $B$ . To that end, we shall need the following result, which is due to [17, Lemma 4.2].

**Proposition 3.4.** *Let  $\Omega, \Omega' \subset \mathbb{R}^n$  be bounded open sets such that  $\Omega \subset\subset \Omega'$ . Let  $A_1, A_2 \in L^\infty(\Omega', \mathbb{C}^n)$ , and  $q_1, q_2 \in L^\infty(\Omega', \mathbb{C})$ . Assume that*

$$A_1 = A_2 \quad \text{and} \quad q_1 = q_2 \quad \text{in} \quad \Omega' \setminus \Omega. \quad (3.23)$$

*If  $C_{A_1,q_1} = C_{A_2,q_2}$  then  $C'_{A_1,q_1} = C'_{A_2,q_2}$ , where  $C'_{A_j,q_j}$  is the set of the Cauchy data for  $L_{A_j,q_j}$  in  $\Omega'$ ,  $j = 1, 2$ .*

*Proof.* Let  $u'_1 \in H^1(\Omega')$  be a solution to  $L_{A_1,q_1}u'_1 = 0$  in  $\Omega'$  and let  $u_1 = u'_1|_\Omega \in H^1(\Omega)$ . As  $C_{A_1,q_1} = C_{A_2,q_2}$ , there exists  $u_2 \in H^1(\Omega)$  satisfying  $L_{A_2,q_2}u_2 = 0$  in  $\Omega$  such that

$$Tu_2 = Tu_1 \quad \text{and} \quad N_{A_2,q_2}u_2 = N_{A_1,q_1}u_1 \quad \text{in} \quad \Omega.$$

In particular,  $\varphi := u_2 - u_1 \in H_0^1(\Omega) \subset H_0^1(\Omega')$ . We define

$$u'_2 = u'_1 + \varphi \in H^1(\Omega'),$$

so that  $u'_2 = u_2$  on  $\Omega$ . It follows that  $Tu'_2 = Tu'_1$  in  $\Omega'$ .

Let us show now that  $L_{A_2,q_2}u'_2 = 0$  in  $\Omega'$ . To that end, let  $\psi \in C_0^\infty(\Omega')$ , and write

$$\begin{aligned} \langle L_{A_2,q_2}u'_2, \psi \rangle_{\Omega'} &= \int_{\Omega'} \left( (\nabla u'_1 + \nabla \varphi) \cdot \nabla \psi + A_2 \cdot (Du'_1 + D\varphi)\psi \right) dx \\ &\quad + \int_{\Omega'} \left( -A_2(u'_1 + \varphi) \cdot D\psi + (A_2^2 + q_2)(u'_1 + \varphi)\psi \right) dx. \end{aligned}$$

Using (3.23), we have

$$\begin{aligned} \langle L_{A_2,q_2}u'_2, \psi \rangle_{\Omega'} &= \int_{\Omega} (\nabla u_2 \cdot \nabla \psi + A_2 \cdot (Du_2)\psi - A_2u_2 \cdot D\psi + (A_2^2 + q_2)u_2\psi) dx \\ &\quad + \int_{\Omega' \setminus \Omega} (\nabla u'_1 \cdot \nabla \psi + A_1 \cdot (Du'_1)\psi - A_1u'_1 \cdot D\psi + (A_1^2 + q_1)u'_1\psi) dx \\ &\quad + \int_{\Omega' \setminus \Omega} (\nabla \varphi \cdot \nabla \psi + A_1 \cdot (D\varphi)\psi - A_1\varphi \cdot D\psi + (A_1^2 + q_1)\varphi\psi) dx. \end{aligned}$$



As  $\varphi \in H_0^1(\Omega)$ , we get

$$\int_{\Omega' \setminus \Omega} (\nabla \varphi \cdot \nabla \psi + A_1 \cdot (D\varphi)\psi - A_1 \varphi \cdot D\psi + (A_1^2 + q_1)\varphi\psi) dx = 0.$$

This together with the fact  $N_{A_2, q_2} u_2 = N_{A_1, q_1} u_1$  in  $\Omega$  implies that

$$\begin{aligned} \langle L_{A_2, q_2} u'_2, \psi \rangle_{\Omega'} &= (N_{A_2, q_2} u_2, [\psi|_{\Omega}])_{\Omega} \\ &+ \int_{\Omega' \setminus \Omega} (\nabla u'_1 \cdot \nabla \psi + A_1 \cdot (Du'_1)\psi - A_1 u'_1 \cdot D\psi + (A_1^2 + q_1)u'_1\psi) dx \\ &= \langle L_{A_1, q_1} u'_1, \psi \rangle_{\Omega'} = 0, \end{aligned}$$

which shows that  $L_{A_2, q_2} u'_2 = 0$  in  $\Omega'$ .

Arguing similarly, we see that  $N_{A_2, q_2} u'_2 = N_{A_1, q_1} u'_1$  in  $\Omega'$ , which allows us to conclude that  $C'_{A_1, q_1} \subset C'_{A_2, q_2}$ . The same argument in the other direction gives the claim.  $\square$

Let us extend  $q_j$ ,  $j = 1, 2$ , to the open ball  $B$  by defining  $q_j = 0$  in  $B \setminus \Omega$ . Then using Proposition 3.4, Lemma 3.1 and the fact that  $\psi|_{\partial B} = 0$ , we obtain that

$$C'_{A_1, q_1} = C'_{A_2, q_2} = C'_{A_2 + \nabla \psi, q_2} = C'_{A_1, q_2}.$$

This implies the following integral identity,

$$\int_B (q_1 - q_2) u_1 \overline{u_2} dx = 0, \quad (3.24)$$

valid for any  $u_1, u_2 \in H^1(B)$  satisfying  $L_{A_1, q_1} u_1 = 0$  in  $B$  and  $L_{\overline{A_1}, \overline{q_2}} u_2 = 0$  in  $B$ , respectively.

Let us choose  $u_1$  and  $u_2$  to be the complex geometric optics solutions in  $B$ , given by (3.6) and (3.10), respectively. In this case, it follows from (3.16) that  $\Phi_1^\sharp(\cdot, \mu_1 + i\mu_2; h) + \Phi_2^\sharp(\cdot, -\mu_1 + i\mu_2; h)$  converges to zero in  $L_{\text{loc}}^2(\mathbb{R}^n)$  as  $h \rightarrow 0$ .

Plugging  $u_1$  and  $u_2$  into (3.24) gives

$$\int_B (q_1 - q_2) e^{ix \cdot \xi} e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} dx = - \int_B (q_1 - q_2) e^{ix \cdot \xi} (e^{\Phi_1^\sharp} \overline{r_2} + r_1 e^{\overline{\Phi_2^\sharp}} + r_1 \overline{r_2}) dx.$$

Letting  $h \rightarrow 0$ , and using (3.7), (3.9), (3.11), and (3.13), we get

$$\int_B (q_1 - q_2) e^{ix \cdot \xi} dx = 0,$$

and therefore,  $q_1 = q_2$  in  $\Omega$ . The proof of Theorem 1.1 is complete.

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## REFERENCES

- [1] Astala, K., Päiväranta, L., *Calderón's inverse conductivity problem in the plane*, Ann. of Math. (2) **163** (2006), no. 1, 265–299.
- [2] Brown, R., *Global uniqueness in the impedance-imaging problem for less regular conductivities*, SIAM J. Math. Anal. **27** (1996), no. 4, 1049–1056.
- [3] Brown, R., Torres, R., *Uniqueness in the inverse conductivity problem for conductivities with  $3/2$  derivatives in  $L^p$ ,  $p > 2n$* , J. Fourier Anal. Appl. **9** (2003), no. 6, 563–574.
- [4] Calderón, A., *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), 65–73, Soc. Brasil. Mat., Rio de Janeiro, 1980.
- [5] Dos Santos Ferreira, D., Kenig, C., Sjöstrand, J., and Uhlmann, G., *Determining a magnetic Schrödinger operator from partial Cauchy data*, Comm. Math. Phys. **271** (2007), no. 2, 467–488.
- [6] Edmunds, D., Evans, W., *Spectral theory and differential operators*, Oxford University Press, New York, 1987.
- [7] Eskin, G., Ralston, J., *Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy*, Comm. Math. Phys. **173** (1995), no. 1, 199–224.
- [8] Greenleaf, A., Lassas, M., and Uhlmann, G., *The Calderón problem for conormal potentials. I. Global uniqueness and reconstruction*, Comm. Pure Appl. Math. **56** (2003), no. 3, 328–352.
- [9] Haberman, B., Tataru, D., *Uniqueness in Calderón's problem with Lipschitz conductivities*, preprint 2011, <http://arxiv.org/abs/1108.6068>.
- [10] Hörmander, L., *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, Classics in Mathematics. Springer-Verlag, Berlin, 2003.
- [11] Kenig, C., Sjöstrand, J., and Uhlmann, G., *The Calderón problem with partial data*, Ann. of Math. (2) **165** (2007), no. 2, 567–591.
- [12] McLean, W., *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.
- [13] Nakamura, G., Sun, Z., and Uhlmann, G., *Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field*, Math. Ann. **303** (1995), no. 3, 377–388.
- [14] Panchenko, A., *An inverse problem for the magnetic Schrödinger equation and quasi-exponential solutions of nonsmooth partial differential equations*, Inverse Problems **18** (2002), no. 5, 1421–1434.
- [15] Päiväranta, L., Panchenko, A., and Uhlmann, G., *Complex geometrical optics solutions for Lipschitz conductivities*, Rev. Mat. Iberoamericana **19** (2003), no. 1, 57–72.
- [16] de Rham, G., *Differentiable manifolds. Forms, currents, harmonic forms*, Grundlehren der Mathematischen Wissenschaften, **266**. Springer-Verlag, Berlin, 1984.
- [17] Salo, M., *Inverse problems for nonsmooth first order perturbations of the Laplacian*, Ann. Acad. Sci. Fenn. Math. Diss. **139** (2004).
- [18] Salo, M., *Semiclassical pseudodifferential calculus and the reconstruction of a magnetic field*, Comm. Partial Differential Equations **31** (2006), no. 10–12, 1639–1666.

- [19] Salo, M., Tzou, L., *Carleman estimates and inverse problems for Dirac operators*, Math. Ann. **344** (2009), no. 1, 161–184.
- [20] Sun, Z., *An inverse boundary value problem for Schrödinger operators with vector potentials*, Trans. Amer. Math. Soc. **338** (1993), no. 2, 953–969.
- [21] Sylvester, J., Uhlmann, G., *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. (2) **125** (1987), no. 1, 153–169.
- [22] Tolmasky, C., *Exponentially growing solutions for nonsmooth first-order perturbations of the Laplacian*, SIAM J. Math. Anal. **29** (1998), no. 1, 116–133.

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